The Poisson Summation Formula

Suppose $f \in C(\mathbb{R}^n)$ satisfies

$$|f(x)| \le C(1+|x|)^{-n-\varepsilon}, |\hat{f}(\xi)| \le C(1+|\xi|)^{-n-\varepsilon}$$

for some $C > 0, \varepsilon > 0$, where $\hat{f}$ is the Fourier transform of f i.e.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \ (\xi \in \mathbb{R}^n).$$

Then

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa\cdot x},$$

where both series converge absolutely and uniformly on n-torus $\mathbb{T}^n$ $\left(\simeq [-\frac{1}{2}, \frac{1}{2})^n\right)$. In particular, taking $x = 0$, we obtain a formula

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa).$$

Proof. Since $\displaystyle\int_{\mathbb{R}^n} (1+|x|)^{-n-\varepsilon}\, dx$ converges, so series $\displaystyle\sum_{k\in\mathbb{Z}^n} (1+|k|)^{-n-\varepsilon} < \infty$ does. Hence, series $\displaystyle\sum_{k\in\mathbb{Z}^n} f(x+k)$ and $\displaystyle\sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa\cdot x}$ converge absolutely and uniformly.

Let Pf, p_κ be

$$Pf(x) = \sum_{k\in\mathbb{Z}^n} f(x+k), p_\kappa = \int_{\mathbb{T}^n} Pf(x) e^{-2\pi i \kappa\cdot x}\, dx.$$

Then p_κ is a κ-th Fourier coefficient of Pf and

$$\begin{aligned}
p_\kappa &= \int_{\mathbb{T}^n} Pf(x) e^{-2\pi i \kappa\cdot x}\, dx \\
&= \int_{\mathbb{T}^n} \sum_{k\in\mathbb{Z}^n} f(x+k) e^{-2\pi i \kappa\cdot x}\, dx \\
&= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n} f(x+k) e^{-2\pi i \kappa\cdot x}\, dx \ (\because \text{uniform convergence}) \\
&= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n+k} f(x) e^{-2\pi i \kappa\cdot (x-k)}\, dx \\
&= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \kappa\cdot x}\, dx \ (\because k\cdot\kappa \in \mathbb{Z}) \\
&= \hat{f}(\kappa).
\end{aligned}$$

Then by Fourier series of Pf, the formulas

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa\cdot x}, \sum_{k\in\mathbb{Z}^n} f(k) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa) \ (x = 0).$$

are proved. ■